Phase synchronization from noisy univariate signals

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We present methods for detecting phase synchronization of two unidirectionally coupled, self-sustained noisy oscillators from a signal of the driven oscillator alone. One method detects soft, another hard phase locking. Both are applied to the problem of detecting phase synchronization in von Kármán vortex flow meters.

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Phase synchronization of nonlinear, self-sustained oscillators is known since the observations of Huygens over 300 years ago [1]. For several reasons, interest in the phenomenon has recently resurged, for review, see [2, 3, 4]. For chaotic oscillators, phase synchronization was found to be an independent regime among various others with different degrees of synchronization [5, 6, 7, 8]. Phase synchronization has also been observed in several natural phenomena [2], most notably perhaps in brain physiology [9, 10, 11], where it is thought to play a key role in information processing [11]. Therefore, interest has risen in the problem of identifying and characterizing phase synchronization from measured time series [12].

For the situation that *signals from both oscillators* are available, several techniques have been developed to quantify the degree of phase synchronization between them [9, 13], to determine the coupling direction [14, 15] and the type of coupling [16].

For *univariate signals* from bivariate systems it is sometimes possible to identify and separate oscillatory components of the two oscillators, and to proceed as in the bivariate case [16, 17, 18].

But when a univariate signal contains oscillations from a single oscillator only, the question whether it is phase-synchronized to some other oscillator is much harder to answer. Important progress towards this goal is the method of Janson *et. al.* [19], where the driving oscillator is identified by observing the regular variations it induces in the period of the driven oscillator. Effectively, these observations are used to reconstruct the dynamics of the pair of coupled oscillators in the spirit of Takens' delay embedding. As for all embedding techniques, the method is sensitive to noise, and careful pre-processing of the data is required for noise reduction.

In this Letter, a different method for detecting synchronization in terms of phase locking from univariate time series, based on nonequilibrium phase statistics, is proposed. It applies to situations with unidirectional coupling where only a signal from the driven oscillator is available. Similar to the method of Rosenblum and Pikovsky for the identification of the coupling direction [14], it is not only robust to noise, but requires the dynamics of the driven oscillator to be perturbed by weak internal noise. It turns out that a regime of "hard" phase locking has to be distinguished from a regime of "soft" phase locking, and two different tests have to be used.

An advantage of the tests proposed here is that they cover all frequency ratios $\omega_1 : \omega_2 \approx n : m$ of driven Oscillator 1 to driving Oscillator 2 with small integers n and m. This includes the cases with $n = 1$, where the method described in [19] cannot be applied. In particular for the case of $1 : 1$ phase locking, where a separation of the signals of the oscillators involved is excluded, no test seems to be known so far.

For simplicity, the theory is first described for this important case only. We specify the conditions for a successful detection of phase locking, and apply it to simulated data. Finally, a demonstration of the method for vortex flow meters is reported.

In most experiments not the phase itself but some oscillatory signal $x(t)$ is measured. One possibility to extract the phase $\phi(t)$ from such a signal is to first calculate the Hilbert transform x_H of the signal,

$$x_H(t) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{x(t')}{t - t'}\, dt' \tag{1}$$

where *P.V.* denotes the Cauchy principal value. Then, the *analytical signal*

$$\zeta(t) = x(t) + i x_H(t) = A(t) \exp(i\phi(t)) \tag{2}$$

gives a consistent way to define the phase $\phi(t)$ [3, 18]. By extracting the phase $\phi(t)$, the higher dimensional dynamics of an oscillating system is projected to one dimension.

A minimal model [3] that generally describes the phase dynamics of two unidirectionally coupled, self-sustained oscillators is given by:

$$\dot{\phi}_1 - \omega_1 = -K \sin(\Delta\phi) + \xi_1 \tag{3a}$$

$$\dot{\phi}_2 - \omega_2 = \xi_2 \quad , \tag{3b}$$

where ξ_1 and ξ_2 represent Gaussian noise with

$$\langle \xi_k(t) \xi_l(t') \rangle = 2 D_k \delta_{k,l} \delta(t - t'), \quad k, l = 1, 2 \quad . \tag{3c}$$

Unidirectional coupling of two self-sustained oscillators in the presence of noise is controlled essentially by the four characteristic time scales of the phase dynamics, which

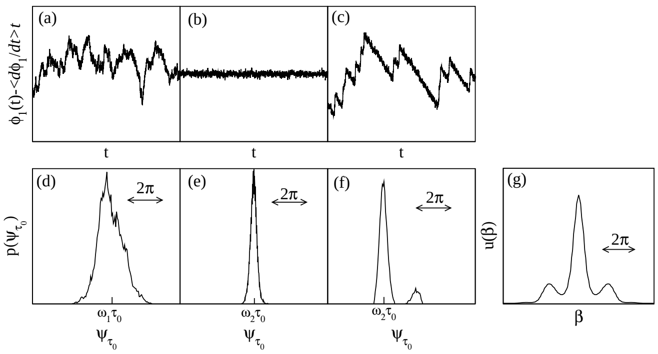

FIG. 1: Typical, mean drift corrected, phase $\phi_1(t)-\langle d\phi_1/dt\rangle\, t$ simulated with equation [3] for (a) no, (b) hard and (c) soft phase locking and the distributions of the phase increment ψ_{τ_0} for each case (d)-(f). (g) the distribution $u(\beta)$, cf. Eq. (8), corresponding to (f).

correspond to the four rate constants entering the minimal model. These are the detuning $\Delta\omega = \omega_1 - \omega_2$ between the oscillators, the diffusion coefficients D_1 and D_2 of the phases ϕ_1 and ϕ_2 of the two oscillators in the uncoupled limit, and the relaxation rate K of the relative phase $\Delta\phi(t) := \phi_1(t) - \phi_2(t)$, which measures the strength of the coupling.

Phase locking occurs when $|\Delta\omega| \lesssim K$. The coupling has a noticeable effect on the dynamics of Oscillator 1 only when $D_1 \leq \mathcal{O}(K)$, otherwise its dynamics is blurred by the internal noise.

The minimal model for the phase dynamics, Eq. (3), can exhibit three different types of dynamical behavior:

- If there is no phase locking, $\phi_1(t)$ performs a random walk with a mean drift $\langle d\phi_1/dt\rangle\, t \sim \omega_1 t$, see Fig. 1a.
- If the phase is locked, it can either be *hard-locked*, i.e. it fluctuates around $\langle d\phi_1/dt\rangle\, t \sim \omega_2 t$, see Fig. 1b,
- or it is *soft-locked*, i.e. the fluctuation of $\phi_1(t)$ around $\langle d\phi_1/dt\rangle\, t \sim \omega_2 t$ is intercepted by rapid increases or decreases of 2π, see Fig. 1c. These jumps are called *phase slips*.

These different behaviors only occur for $D_2 \ll D_1$ which is assumed in the following. Otherwise, $\phi_1(t)$ always performs a random walk.

Denoting the average number of phase slips per unit time by r, the regime for soft phase locking is, expressed in the characteristic time scales,

$$D_2 \ll r \ll K \quad , \tag{4}$$

whereas for the regime of hard phase locking it is

$$r \leq \mathcal{O}(D_2) \quad . \tag{5}$$

Based on these properties of the minimal model, the test for phase locking is performed in two steps: A first test checks for the frequent occurrence of phase slips. If the outcome of this test is negative, a second test is performed to distinguish the case of hard phase locking from the absence of synchronization.

In order to perform the first test, the phase increment

$$\psi_\tau(t) := \phi_1(t+\tau) - \phi_1(t) \tag{6}$$

over time intervals of length τ is considered. First, an estimate of the probability distribution function $p(\psi_{\tau_0})$ for some fixed delay τ_0 is determined. Phase slips in $\phi_1(t)$ generally lead to a multimodal distribution $p(\psi_{\tau_0})$ with maxima separated by 2π as displayed in Fig. 1f, if τ_0 is chosen to be longer than the duration of a phase slip $\mathcal{O}(K^{-1})$ [20] and shorter than the coherence time of Oscillator 2:

$$K^{-1} \ll \tau_0 \ll D_2^{-1} \quad . \tag{7}$$

The estimate of $p(\psi_{\tau_0})$ can be rather noisy. In order to obtain a smooth estimator that conserves the possible multimodality of $p(\psi_{\tau_0})$, the auto-covariance function $u(\beta)$ of $p(\psi_{\tau_0})$ is considered:

$$u(\beta) := \int_{-\infty}^{\infty} p(\psi_{\tau_0}) p(\psi_{\tau_0}+\beta)\, d\psi \quad , \tag{8}$$

Multimodality in $p(\psi_{\tau_0})$ translates to multimodality in $u(\beta)$, see Fig. 1g, but the maxima are now located at the positions $\beta = 2\pi k$ ($k \in \mathbb{Z}$).

The degree to which $u(\beta)$ has such a multimodal structure can be quantified by the *raw phase-slip index*

$$\eta_{raw} := \frac{1}{u(0)} \sum_{k=-\infty}^{\infty} u(2k\pi) - u((2k-1)\pi) \quad . \tag{9}$$

In the case of soft phase locking, determined by the relation between the characteristic time scales of the minimal model given in Eq. (4), $u(\beta)$ can be approximated by a sum of Gaussians

$$u(\beta) = \frac{1}{\sqrt{2\pi}\sigma} \sum_l a_l(\tau_0) \exp\left(-\frac{(\beta - 2\pi l)^2}{2\sigma^2}\right) \quad , \tag{10}$$

where $a_l(\tau_0)$ gives the probability that in the time interval τ_0 a net number of l phase slips occur, and σ is the width of the Gaussians. If the internal noise is weak, i.e. $D_1 \ll K$, these Gaussians are distinct and the contribution of the minima $u((2k-1)\pi)$ in Eq. (9) can be neglected. With $\sum_l a_l(\tau_0) = 1$ this leads to an upper bound $\eta_{max} = a_0(\tau_0)^{-1}$ for η_{raw}. Assuming a Poisson distribution for the phase jumps, $a_l(\tau_0)$ can be derived analytically in dependence of the average number of phase slips per unit time r [21]:

$$a_l(\tau_0) = \exp(-2r\tau_0)\, I_l(2r\tau_0) \quad , \tag{11}$$

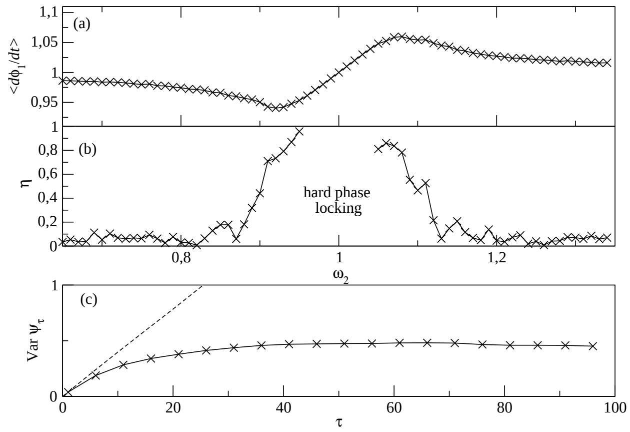

FIG. 2: Simulation results. (a) Observed frequency of driven oscillator in dependence on frequency of driving oscillator. (b) Phase-slip index η in dependence on driving frequency ω_2. (c) Variance of $\mathrm{var}\,\psi_\tau$ *vs.* τ. The saturation of $\mathrm{var}\,\psi_\tau$ indicates hard phase locking at $\omega_2 = 1.0$ (solid) compared to the uncoupled case (dashed).

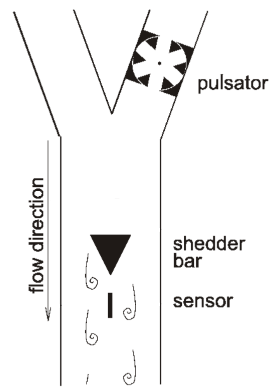

FIG. 3: Experimental setup: A pulsator generates a flow pulsating with frequency f_{puls}. Downstream vortices are formed by a shedder bar. The frequency of the vortices f_{vort} is determined via a piezoelectric sensor.

with $I_l(.)$ denoting the modified Bessel function of the first kind [22]. The time delay τ_0 of the phase increment ψ_{τ_0} has to be chosen large enough that a noticeable number of phase slips occurs. At the same time it has to be small enough that the multimodality of $u(\beta)$ is not blurred by the phase diffusion of the driving oscillator. Numerical simulations show that choosing τ_0 such that the Gaussian peak at the origin contributes $\approx 75\%$ to $u(\beta)$ is a good choice. With $a_0(\tau_0) = 0.75$, it follows $\eta_{max} = 1.33$. Analytical calculations [21] reveal that in the case of soft locking the value of $\mathrm{var}(\beta)$, with β distributed as $u(\beta)$, can be estimated as $8\pi^2 r\tau_0$. Thus from fixing $a_l(\tau_0)$ we obtain

$$\mathrm{var}(\beta) = 12.2 \quad . \tag{12}$$

Since the actual value of r is not known, Eq. (12) is used as the condition for choosing τ_0.

By definition the lowest conceivable value for η_{raw} is obtained when $p(\psi_{\tau_0})$ and consequently $u(\beta)$ are single Gaussians. With the fixed variance of $u(\beta)$ this gives a minimal value $\eta_{min} = 0.014$ [21] for η_{raw} and a *normalized phase-slip index* η can be defined as

$$\eta := \frac{\eta_{raw} - \eta_{min}}{\eta_{max} - \eta_{min}} \quad , \tag{13}$$

which takes values between 0 and 1 depending on how well pronounced phase slips are. Large values indicate soft phase locking. In the case of hard phase locking, no τ_0 can be determined such that Eq. (12) holds, and η cannot be calculated.

If the test for soft phase locking is negative, a second test has to be performed. Absence of phase slips can have two reasons: Either oscillator 1 is not synchronized to oscillator 2 at all or hard phase locking is present. As a means to decide between the two alternatives, the dependence of $\mathrm{var}\,\psi_\tau$ on τ is considered. For hard phase locking, the phase $\phi_1(t)$ is described by a stochastic process trapped in a local minimum of the 2π-periodic interaction potential of the two oscillators [23]. In the simplest case this is an Ornstein-Uhlenbeck process [23] with $\mathrm{cov}[\phi_1(t), \phi_1(t+\tau)] = (D/K)\exp(-\tau K)$, where $D = D_1 + D_2 \approx D_1$. It follows that two regions in the dependence of $\mathrm{var}\,\psi_\tau$ on τ can be distinguished. For small τ there is an approximately linear increase with $\mathrm{var}\,\psi_\tau \approx 2D_1\tau$. This saturates at $\tau \approx K^{-1}$ and, for larger τ, $\mathrm{var}\,\psi_\tau = 2D_1/K$ is constant. Such a saturation is the signature of hard phase locking. If oscillator 1 is not synchronized at all, $\phi_1(t)$ performs a random walk, and the variance of ψ_τ grows linearly in τ with slope $2D_1$ [24].

When the test for soft phase locking is applied to the general $n : m$ case, the phase $\phi_1(t)$ obtained from the signal has to be multiplied by m prior to computing ψ_{τ_0}. In the test for hard phase locking, no modifications are required.

The proposed procedure is exemplified by a simulation study based on the minimal model, Eq. (3), with parameters $K = 0.1$, $D_1 = 0.02$, $D_2 = 0$, $\omega_1 = 1$, and ω_2 varied between 0.65 and 1.35. Fig. 2a shows the observed frequency $\langle d\phi_1/dt\rangle$ in dependence on ω_2, Fig. 2b the normalized phase-slip index η in dependence on ω_2, calculated from samples of $\phi_1(t)$ of length $N = 10^6$ with 10 samples per unit time. For ω_2 between 0.96 and 1.04, η cannot be calculated since the condition given in Eq. (12) cannot be fulfilled. Fig. 2c displays the dependence of the variance of ψ_τ on τ. For the hard phase locked case at $\omega_2 = 1$, ψ_τ saturates for large values of τ, while for the uncoupled case with $K = 0$, the variance increases linearly.

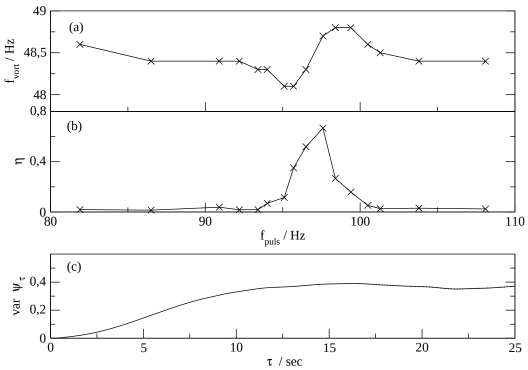

FIG. 4: Experimental results. (a) Vortex frequency and (b) phase-slip index η in dependence on pulsation frequency f_{puls}. (c) The saturation of $\operatorname{var}\psi_\tau$ *vs.* τ within the region of hard phase locking at a larger pulsation amplitude.

We apply the proposed procedure to detect phase locking in vortex flow meters that are applied to non-invasively measure flow velocity of fluids [25]. These devices make use of a von Kármán vortex street that forms behind a shedder bar inserted orthogonal to the flow in a pipe. The velocity of the fluid can be determined from the frequency of vortex-pair formation f_{vort}. In the device used here, a piezoelectric sensor inserted downstream behind the shedder bar is used to detect the pressure oscillations generated from vortices passing by. A common problem of vortex flowmetering is phase locking of the vortices to pulsatile flow.

In the experiments, pulsations with frequency f_{puls} were generated by adding a periodically blocked flow to a steady flow. The degree of flow modulation could be adjusted by the relative contribution of the two, see Fig. 3. In a first experiment, the peak-to-peak flow modulation was $\approx 10\%$ of the average flow rate, and the frequency of the flow pulsation f_{puls} was varied. Due to the symmetry of the setup [18], the strongest phase locking occurs at a frequency ratio of $f_{\text{puls}} : f_{\text{vort}} = 2 : 1$. In Fig. 4a, the shift in f_{vort} due to phase locking at $f_{\text{puls}} \approx 2\, f_{\text{vort}}$ is clearly visible.

The normalized phase-slip index η, Eq. (13), was computed from 65 s time series of the sensor signal. As shown in Fig. 4b, the presence of phase locking is indicated by a pronounced rise in η. Phase locking was always soft. In a second experiment, pulsation was increased to $\approx 40\%$. The condition to calculate η were not fulfilled. Figure 4c displays the $\operatorname{var}\psi_\tau$ in dependence on τ. The saturation for large values of τ indicates hard phase locking.

In summary, based on a minimal model for the phase dynamics of unidirectionally coupled oscillators, we proposed a two-step procedure that provides sensitive and specific indicators for soft, hard, and no phase locking. Soft phase locking is detected by the presence of phase slips, hard phase locking by a suppression of phase diffusion, while no phase locking is indicated by divergent phase diffusion. Both methods were demonstrated on experimental data from vortex flow meters.

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[1] C. Huygens, *The Pendulum Clock* (Iowa State University Press, Ames, 1986 (Latin Original: 1673)).
[2] S. Boccaletti *et al.*, Physics Reports **366**, 1 (2002).
[3] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization - A Universal Concept in Nonlinear Sciences*, *Cambrige Nonlinear Science Series* (Cambridge University Press, Cambridge, 2001).
[4] E. Mosekilde, Y. Maistrenko, and D. Postnov, *Chaotic Synchronization: Application to Living Systems* (World Scientific, Singapore, 2002).
[5] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).
[6] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **78**, 4193 (1997).
[7] K. Josić and D. J. Mar, Phys. Rev. E **64**, 056234 (2001).
[8] F. R. Drepper, Phys. Rev. E **62**, 6376 (2000).
[9] P. Tass *et al.*, Phys. Rev. Lett. **81**, 3291 (1998).
[10] J. Bhattacharya and H. Petsche, Phys. Rev. E **64**, 012902 (2001).
[11] F. Varela, J. Lachaux, E. Rodriguez, and J. Martinerie, Nature Rev. Neuroscience **2**, 229 (2001).
[12] M. Rosenblum *et al.*, in *Neuro-Informatics*, Vol. 4 of *Handbook of Biological Physics*, edited by F. Moss and S. Gielen (Elsevier Science, Amsterdam, 2001), Chap. 9, pp. 279–321.
[13] C. Allefeld and J. Kurths, Int. J. Bif. Chaos **14**, 405 (2004).
[14] M. G. Rosenblum and A. S. Pikovsky, Phys. Rev. E **64**, 045202(R) (2001).
[15] M. G. Rosenblum *et al.*, Phys. Rev. E **65**, 041909 (2002).
[16] J. Jamšek, A. Stefanovska, P. V. E. McClintock, and I. A. Khovanov, Phys. Rev. E **68**, 016201 (2003).
[17] A. Stefanovska and M. Hozic, Prog. of Theor. Phys.(Suppl.) **139**, 270 (2000).
[18] A. G. Rossberg, K. Bartholome, and J. Timmer, Phys. Rev. E **69**, 016216 (2004).
[19] N. B. Janson, A. G. Balanov, V. S. Anishchenko, and P. V. E. McClintock, Phys. Rev. Lett. **86**, 1749 (2001).
[20] R. L. Stratonovich, in *Topics in the Theory of Random Noise*, edited by R. L. Stratonovich (Gordon and Breach, New York, 1967).
[21] A. G. Rossberg, K. Bartholome, H. U. Voss, and J. Timmer (unpublished).
[22] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (Dover, New York, 1972).
[23] H. Risken, *The Fokker-Planck Equation* (Spriner Verlag, Berlin, 1989).
[24] A. Einstein, Ann. Phys. **17**, 549 (1905).
[25] K. Al-Asmi and P. Castro, Flow Meas. Instrum. **3**, 187 (1992).